

Assignment 3: Limits and Colimits

CS 7480: Categories for PL, Fall 2025

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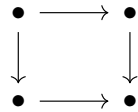
Due November 14

Problem 1 (Warming up to categorical logic). In this exercise you will examine limits and colimits in the category *Formula* of Boolean formulae from Homework 1. Recall that the objects of this category are formulas

$$\varphi, \psi ::= x \in \mathbf{Var} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg \varphi$$

where \mathbf{Var} is a countably infinite set of variables, and there is a morphism from φ to ψ in this category if and only if $\text{models}(\varphi) \subseteq \text{models}(\psi)$.

1. In Homework 1 you showed that *Formula* has finite products (the terminal object is the always-true formula $x \vee \neg x$, and binary products are given by \wedge). Does *Formula* also have equalizers? If so, what do they look like?
2. Describe what finite coproducts and coequalizers look like in *Formula*.

3. Give an example of a limit and colimit for a diagram of shape  in *Formula*. You can pick whichever diagram you like.

4. Consider the infinite diagram

$$x \longrightarrow x \vee y \longrightarrow x \vee y \vee z \longrightarrow \dots$$

in *Formula*. Does this diagram have a colimit?

Problem 2 (A limit here, a limit there). Let $D : \mathcal{I} \rightarrow \mathcal{C}$ be a diagram of shape \mathcal{I} in a category \mathcal{C} , and $F : \mathcal{C} \rightarrow \mathcal{D}$ a functor. We say that

- F *preserves limits of shape D* if, whenever (P, p) is a limiting cone over D in the category \mathcal{C} , it holds that $(FP, (F(p_i) : F(P) \rightarrow F(D(i)))_{i \in \mathcal{I}})$ is a limiting cone over $F \circ D$ in the category \mathcal{D} .
- F *reflects limits of shape D* if, whenever (P, p) is a cone over D in \mathcal{C} such that $(FP, (F(p_i) : F(P) \rightarrow F(D(i)))_{i \in \mathcal{I}})$ is limiting in \mathcal{D} , it holds that (P, p) is limiting in \mathcal{C} .
- F *creates limits of shape D* if it reflects limits of shape D and moreover, whenever (Q, q) is a limiting cone over $F \circ D$ in \mathcal{D} , there exists a limiting cone (P, p) over D in \mathcal{C} such that $Q = F(P)$ and $q_i = F(p_i)$ for all $i \in \mathcal{I}$.

Show that

1. If F is full and faithful, then it reflects limits of any shape.
2. If F is a left adjoint, then it preserves colimits of any shape.
3. If $F \circ D$ has a limit in \mathcal{D} , and F creates limits of shape D , then \mathcal{C} has all limits of shape D too.
4. If F creates limits of shape D , and $F \circ D$ has a limit in \mathcal{D} , then F also preserves limits of shape D .

Problem 3 (Suddenly, logical relations). Given two functors $\mathcal{C} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{D}$, the *comma category* of F and G , written $F \downarrow G$ or sometimes as (F, G) , is the category whose

- Objects are tuples (c, d, p) where c is an object of \mathcal{C} , d is an object of \mathcal{D} , and p is a morphism $Fc \rightarrow Gd$ in \mathcal{B}
- Morphisms from (c, d, p) to (c', d', p') are pairs (f, g) where $c \xrightarrow{f} c'$ and $d \xrightarrow{g} d'$ are morphisms in \mathcal{C} and \mathcal{D} respectively such that the following square commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{Ff} & Fc' \\ p \downarrow & & \downarrow p' \\ Gd & \xrightarrow{Gg} & Gd' \end{array}$$

1. Show that the category of cones over a diagram $D : \mathcal{I} \rightarrow \mathcal{C}$ is isomorphic to the comma category $F \downarrow G$ where $F : \mathcal{C} \rightarrow [\mathcal{I}; \mathcal{C}]$ and $G : 1 \rightarrow [\mathcal{I}; \mathcal{C}]$ are defined on objects by $F(c) = \Delta(c)$ and $G(\star) = D$. (As part of your answer, you will have to fill in the definitions of F and G on morphisms.)
2. Specialize the definition of comma category to the case where
 - $\mathcal{B} = \mathcal{C} = \text{Set}$
 - \mathcal{D} is a category with a terminal object 1
 - F is the identity functor $\text{id}_{\text{Set}} : \text{Set} \rightarrow \text{Set}$
 - G is the functor $\mathcal{D}(1, -) : \mathcal{D} \rightarrow \text{Set}$

The corresponding comma category $\text{id}_{\text{Set}} \downarrow \mathcal{D}(1, -)^1$ is called the “scone” over \mathcal{D} , and is sometimes written $\text{Scn}(\mathcal{D})$. Show that this category has a terminal object, and that it has products and coproducts if \mathcal{D} does.

3. Define a functor $P : \text{Scn}(\mathcal{D}) \rightarrow \text{Set}$ such that the previous bullet point is equivalent to saying that P creates terminal objects, products, and coproducts.
4. Consider the special case $\mathcal{D} = \text{STLC}$, the category whose objects are STLC types and whose morphisms are terms with one free variable. Using the previous bullet points, deduce that $\text{Scn}(\text{STLC})$ has a terminal object, products, and coproducts. In what sense do the definitions of these universal constructions resemble logical relations?
5. Can you find a way to define exponentials in $\text{Scn}(\text{STLC})$?

¹Recall the notation $\mathcal{D}(X, -)$ denotes the set of morphisms out of some object X in \mathcal{C}