Assignment 3: Limits and Colimits

CS 7480: Categories for PL, Fall 2025
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Problem 1 (Warming up to categorical logic)**.** In this exercise you will examine limits and colimits in the category $\mathsf{Formula}$ of Boolean formulae from Homework 1. Recall that the objects of this category are formulas

$$\varphi, \psi ::= x \in \mathsf{Var} \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\varphi$$

where Var is a countably infinite set of variables, and there is a morphism from φ to ψ in this category if and only if $\mathsf{models}(\varphi) \subseteq \mathsf{models}(\psi)$.

1. In Homework 1 you showed that $\mathsf{Formula}$ has finite products (the terminal object is the always-true formula $x \vee \neg x$, and binary products are given by $\wedge$). Does $\mathsf{Formula}$ also have equalizers? If so, what do they look like?
2. Describe what finite coproducts and coequalizers look like in $\mathsf{Formula}$.
3. Give an example of a limit and colimit for a diagram of shape $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$ in $\mathsf{Formula}$. You can pick whichever diagram you like.
4. Consider the infinite diagram

$$x \longrightarrow x \vee y \longrightarrow x \vee y \vee z \longrightarrow \ldots$$

in $\mathsf{Formula}$. Does this diagram have a colimit?

Problem 2 (A limit here, a limit there). Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram of shape $\mathcal{I}$ in a category $\mathcal{C}$, and $F : \mathcal{C} \to \mathcal{D}$ a functor. We say that

- *F preserves limits of shape D* if, whenever (P, p) is a limiting cone over D in the category $\mathcal{C}$, it holds that $(FP, (F(p_i) : F(P) \to F(D(i)))_{i \in \mathcal{I}})$ is a limiting cone over $F \circ D$ in the category $\mathcal{D}$.
- *F reflects limits of shape D* if, whenever (P, p) is a cone over D in $\mathcal{C}$ such that $(FP, (F(p_i) : F(P) \to F(D(i)))_{i \in \mathcal{I}})$ is limiting in $\mathcal{D}$, it holds that (P, p) is limiting in $\mathcal{C}$.
- *F creates limits of shape D* if it reflects limits of shape D and moreover, whenever (Q, q) is a limiting cone over $F \circ D$ in $\mathcal{D}$, there exists a limiting cone (P, p) over D in $\mathcal{C}$ such that $Q = F(P)$ and $q_i = F(p_i)$ for all $i \in \mathcal{I}$.

Show that

1. If F is full and faithful, then it reflects limits of any shape.
2. If F is a left adjoint, then it preserves colimits of any shape.
3. If $F \circ D$ has a limit in $\mathcal{D}$, and F creates limits of shape D, then $\mathcal{C}$ has all limits of shape D too.
4. If F creates limits of shape D, and $F \circ D$ has a limit in $\mathcal{D}$, then F also preserves limits of shape D.

Problem 3 (Suddenly, logical relations)**.** Given two functors $\mathcal{C} \xrightarrow{F} \mathcal{B} \xleftarrow{G} \mathcal{D}$, the *comma category of F and G*, written $F \downarrow G$ or sometimes as (F, G), is the category whose

- Objects are tuples (c, d, p) where c is an object of $\mathcal{C}$, d is an object of $\mathcal{D}$, and p is a morphism $Fc \to Gd$ in $\mathcal{B}$
- Morphisms from (c, d, p) to (c', d', p') are pairs (f, g) where $c \xrightarrow{f} c'$ and $d \xrightarrow{g} d'$ are morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively such that the following square commutes:

$$\begin{array}{ccc} Fc & \xrightarrow{Ff} & Fc' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ Gd & \xrightarrow[Gg]{} & Gd' \end{array}$$

1. Show that the category of cones over a diagram $D : \mathcal{I} \to \mathcal{C}$ is isomorphic to the comma category $F \downarrow G$ where $F : \mathcal{C} \to [\mathcal{I}; \mathcal{C}]$ and $G : 1 \to [\mathcal{I}; \mathcal{C}]$ are defined on objects by $F(c) = \Delta(c)$ and $G(\star) = D$. (As part of your answer, you will have to fill in the definitions of F and G on morphisms.)

2. Specialize the definition of comma category to the case where

   - $\mathcal{B} = \mathcal{C} = \mathsf{Set}$
   - $\mathcal{D}$ is a category with a terminal object 1
   - F is the identity functor $\mathsf{id}_{\mathsf{Set}} : \mathsf{Set} \to \mathsf{Set}$
   - G is the functor $\mathcal{D}(1, -) : \mathcal{D} \to \mathsf{Set}$

   The corresponding comma category $\mathsf{id}_{\mathsf{Set}} \downarrow \mathcal{D}(1, -)$[1] is called the "scone" over $\mathcal{D}$, and is sometimes written $\mathsf{Scn}(\mathcal{D})$. Show that this category has a terminal object, and that it has products and coproducts if $\mathcal{D}$ does.

3. Define a functor $P : \mathsf{Scn}(\mathcal{D}) \to \mathsf{Set}$ such that the previous bullet point is equivalent to saying that P creates terminal objects, products, and coproducts.

4. Consider the special case $\mathcal{D} = \mathsf{STLC}$, the category whose objects are STLC types and whose morphisms are terms with one free variable. Using the previous bullet points, deduce that $\mathsf{Scn}(\mathsf{STLC})$ has a terminal object, products, and coproducts. In what sense do the definitions of these universal constructions in resemble logical relations?

5. Can you find a way to define exponentials in $\mathsf{Scn}(\mathsf{STLC})$?

[1]Recall the notation $\mathcal{D}(X, -)$ denotes the set of morphisms out of some object X in $\mathcal{C}$