Assignment 2: Universal Properties

CS 7480: Categories for PL, Fall 2025
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Problem 1 (Monos and epis)**.** It's common in category theory to *categorify* concepts from set theory. Let's see some more examples of this idea: we will categorify the notion of injective and surjective functions. You will show that these correspond to morphisms satisfying some properties:

Definition 1: Monomorphism

A morphism $A \overset{f}{\rightarrowtail} B$ in a category $\mathcal{C}$ is called a *monomorphism* if, for any object Z with morphisms $Z \xrightarrow{g} A$, $Z \xrightarrow{h} B$, it is the case that $f \circ g = f \circ h$ implies $g = h$.

In other words, the monomorphisms in a category are those morphisms that are "cancellable from the left". The dual notion of monorphism is an epimorphism, which are "cancellable from the right":

Definition 2: Epimorphism

A morphism $A \overset{f}{\twoheadrightarrow} B$ is called an *epimorphism* if, for any object Z with morphisms $B \xrightarrow{g} Z$ and $B \xrightarrow{h} Z$, it is the case that $g \circ f = h \circ f$ implies $g = h$.

1. For two finite sets A and B, a function $f : A \to B$ is called *injective* if, for every $a, b \in A$ where $a \neq b$, it is the case that $f(a) \neq f(b)$. Show that, in the category FinSet, a morphism is a monomorphism if and only if it is an injective function.

2. For two finite sets A and B, a function $f : A \to B$ is called *surjective* if, for every $b \in B$, there is an $a \in A$ such that $f(a) = b$. Show that in the category FinSet, a morphism is an epimorphism if and only if it is a surjective function.

3. Give a nontrivial example of an epimorphism and a monomorphism in $\mathsf{FinSet}^{\circlearrowright}$.

4. FinSet satisfies the peculiar property that its morphisms that are both monomorphisms and epimorphisms are isomorphisms. Such a category is called *balanced*. Not all categories are balanced; can you think of an example that isn't?

5. Show that monomorphisms satisfy some nice closure properties:

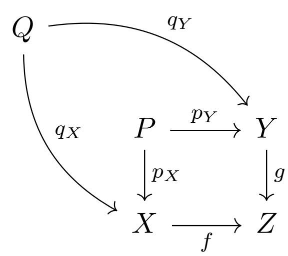
- Identity morphisms are monomorphisms.
- The composition of monomorphisms is a monomorphism.
- If $f \circ g$ is a monomorphism, then so is g.

Then state and prove the dual properties for epimorphisms.

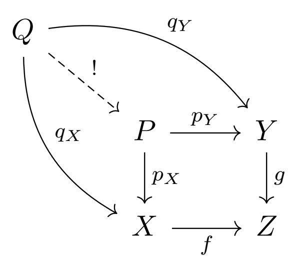
Problem 2 (Pullbacks)**.** Let's see some more examples of universal properties that aren't quite so PL-flavored:

Definition 3: Pullback

For two morphisms $\begin{array}{ccc} & & Y \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$ of a category $\mathcal{C}$, a *pullback of f and g* is a tuple $(P, P \xrightarrow{p_X} X, P \xrightarrow{p_Y} Y)$ for which the diagram $\begin{array}{ccc} P & \xrightarrow{p_Y} & Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Z \end{array}$ commutes, satisfying the universal property that for any other object Q with morphisms $Q \xrightarrow{q_X} X, Q \xrightarrow{q_Y} Y$ such that the diagram:



commutes, there exists a unique morphism $Q \xrightarrow{!} P$ such that this diagram commutes:



1. Show that FinSet has pullbacks. What do they mean in this category?

2. Just like in programming languages, structure can emerge in categories as we combine features of the category. Here's an example: show that if a category has terminal objects and pullbacks, then it has products.[1]

3. What $\mathcal{C}$-indexed set does the pullback of f and g represent? Concretely, identify an isomorphism between $\mathcal{C}(Q, P)$ and X_Q for some $\mathcal{C}$-indexed set X.

4. Show that "the pullback of a monomorphism is a monomorphism": that is, if f is a monomorphism, then the morphism p_Y shown in the squares above is a monomorphism too.

[1]In fact, saying a category has pullbacks and terminal objects is equivalent to saying that it has all finite limits, of which product is an example; we will return to this later.

Problem 3. Given two morphisms $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in a category $\mathcal{C}$, one can define a $\mathcal{C}$-indexed set $\mathsf{Eq}(f,g)$, whose elements intuitively are morphisms that "make f and g equal":

$$\mathsf{Eq}(f,g)_\Gamma = \{\Gamma \xrightarrow{h} X \mid f \circ h = g \circ h\}.$$

Substitution in this $\mathcal{C}$-indexed set acts by composition: given $h \in \mathsf{Eq}(f,g)_\Gamma$ and $\Gamma' \xrightarrow{s} \Gamma$,

$$h \cdot_{\mathsf{Eq}(f,g)} s = h \circ s.$$

A representation for $\mathsf{Eq}(f,g)$ is called an *equalizer of f and g*.

1. Let E be an object of $\mathcal{C}$ and $\alpha : \text{よ}E \Rightarrow \mathsf{Eq}(f,g)$ a representation for $\mathsf{Eq}(f,g)$. Apply the Yoneda lemma to obtain a universal element for α. What does this element look like?

2. Write down the category of elements for $\mathsf{Eq}(f,g)$, and use this to obtain an explicit description of what it means for the universal element you found in part (1) to be terminal in this category.

3. Turn the universal property you found in part (2) into an alternative definition for the equalizer of f and g that does not mention $\mathcal{C}$-indexed sets or representations.

Problem 4 (The category of finite G-sets is Cartesian-closed)**.** Let's see an example of a category with an interesting exponential object and get practice finding exponential objects in this category. Along the way, we'll learn about a new interesting category called the category of G-sets that will be important later (it is one of the the foundational categories for working with nominal sets).

For this problem we need a few auxiliary definitions:

Definition 4: Finite group

A *finite group* is a tuple $(A, \bullet, e)$ where A is a finite set, $\bullet : A \times A \to A$ is a function, and e is an element of A satisfying the group axioms (for $a, b, c \in A$): (1) associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$; (2) identity: $a \bullet e = e \bullet a = a$; (3) inverse: for any $a \in A$, there exists an element $a^{-1} \in A$ such that $a \bullet a^{-1} = a^{-1} \bullet a = e$.

Definition 5: Group actions, finite G-sets

Let A be a finite set and $(G, \bullet, e)$ be a finite group. Then, a function $\cdot : G \times A \to A$ is called a *left action of G on A* if for any $g_1, g_2 \in G$ and $a \in A$, it holds that $g_1 \cdot (g_2 \cdot a) = (g_1 \bullet g_2) \cdot a$. The pair $(A, \cdot)$ is called a *finite G-set*.

Now, we can define the category FinSet_G of finite G-sets where:

- Objects are finite G-sets $(A, \cdot_A)$ where $\cdot_A$ is the action of the group G on the set A.
- Morphisms $(A, \cdot_A) \xrightarrow{f} (B, \cdot_B)$ are G-equivariant functions, i.e., they are functions $f : A \to B$ satisfying that for any $a \in A$ and $g \in G$, it is the case that $g \cdot_B f(a) = f(g \cdot_A a)$.
- The identity morphism is the identity function.
- Composition of morphisms is function composition.

1. Show FinSet_G is a category.
2. Show FinSet_G is Cartesian-closed (*Note*: this is hard, don't spend too much time on it).

Problem 5 (The category of pointed sets $\mathsf{FinSet}^{\star}$)**.** The category of pointed sets $\mathsf{FinSet}^{\star}$ is a category where:

- Objects are pairs (A, a_0) where A is a finite set and a_0 is a distinguished element of A;
- Morphisms $(A, a_0) \xrightarrow{f} (B, b_0)$ are functions satisfying $f(a_0) = b_0$, i.e., they preserve distinguished elements.

1. Show that $\mathsf{FinSet}^{\star}$ is a category.
2. Show that $\mathsf{FinSet}^{\star}$ has finite products and a terminal object.
3. Show that terminal objects and initial objects are the same in this category.
4. Show that $\mathsf{FinSet}^{\star}$ *does not* have all exponential objects.