Assignment 1

CS 7480: Categories for PL, Fall 2025
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Due Friday, Sept 26

Problem 1 (Formula: the category of syntactic Boolean formulae and entailment)**.** Consider the following grammar of Boolean formulae:

$$\varphi, \psi ::= x \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \neg\psi \tag{1}$$

Fix a global finite set of variable names Ω, and assume $x \in \Omega$. Then, we can give a semantics to Boolean formulae in terms of substitutions. A substitution γ is a function $\Omega \to \{\top, \bot\}$ that maps each variable to a truth value. Then, we can define the semantics of formulae inductively:

$$\begin{aligned}
[\![x]\!](\gamma) &= \gamma(x) \\
[\![\varphi \wedge \psi]\!](\gamma) &= \begin{cases} \top & \text{if } [\![\varphi]\!](\gamma) = \top, [\![\psi]\!](\gamma) = \top \\ \bot & \text{otherwise} \end{cases} \\
[\![\varphi \vee \psi]\!](\gamma) &= \begin{cases} \bot & \text{if } [\![\varphi]\!](\gamma) = \bot, [\![\psi]\!](\gamma) = \bot \\ \top & \text{otherwise} \end{cases} \\
[\![\neg\varphi]\!](\gamma) &= \begin{cases} \top & \text{if } [\![\varphi]\!](\gamma) = \bot \\ \bot & \text{if } [\![\varphi]\!](\gamma) = \top \end{cases}
\end{aligned}$$

The *set of models*, written $\mathsf{Mods}(\varphi)$, is the set of all substitutions that evaluate to $\top$, i.e. $\mathsf{Mods}(\varphi) = \{\gamma \mid [\![\varphi]\!](\gamma) = \top\}$. We say a formula φ *semantically entails* a formula ψ, written $\varphi \models \psi$, if $\mathsf{Mods}(\varphi) \subseteq \mathsf{Mods}(\psi)$.

Consider the following components of a category whose objects are Boolean formula and whose morphisms denote formula entailment:

- Let the set of objects O be the set of all syntactic Boolean formulae generated by the above grammar.
- The set of morphisms M consists of pairs (φ, ψ) where $\varphi \models \psi$.
- Define $\mathsf{dom}(\varphi, \psi) = \varphi, \mathsf{cod}(\varphi, \psi) = \psi$
- Identity is defined by $\mathsf{id}(\phi) = (\phi, \phi)$
- Composition is defined by $\mathsf{comp}((\psi, \xi), (\varphi, \psi)) = (\varphi, \xi)$

Verify that the above satisfy the definition of a category. This will involve showing that some entailments are valid.

Problem 2 ($\mathsf{FinSet}^{\circlearrowleft}$: the category of finite transition systems)**.** Consider the following components of a category whose objects are finite transition systems and whose morphisms denote "abstractions" of transition systems:

- Objects are pairs (X, α) where X is a finite set and $\alpha : X \to X$ is a function.
- Morphisms are triples $((X, \alpha), f, (Y, \beta))$ where (1) f is a function $f : X \to Y$, and (2) f satisfies $f \circ \alpha = \beta \circ f$. These morphisms can be drawn as commutative squares:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{f} & Y \end{array}$$

- The domain of a morphism is:

$$\mathsf{dom}\left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{f} & Y \end{array}\right) = (X, \alpha)$$

- The codomain of a morphism is:

$$\mathsf{dom}\left(\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ X & \xrightarrow{f} & Y \end{array}\right) = (Y, \beta)$$

- The identity morphism is $\mathsf{id}((X, \alpha)) = ((X, \alpha), x \mapsto x, (X, \alpha))$ where $x \mapsto x$ denotes the identity function on X.
- For two morphisms $((X, \alpha), f, (Y, \beta))$ and $((Y, \beta), g, (Z, \gamma))$, composition is defined in terms of composition of functions:

$$((Y, \beta), g, (Z, \gamma)) \circ ((X, \alpha), f, (Y, \beta)) = ((X, \alpha), g \circ f, (Z, \gamma))$$

1. Show that $\mathsf{FinSet}^{\circlearrowleft}$ satisfies the definition of a category. This will entail showing that id and comp define valid morphisms.
2. Draw some internal pictures of morphisms in this category.
3. In what sense does a morphism $((X, \alpha), f, (Y, \beta))$ define an abstraction of the transition system (X, α)?

Problem 3 (The slice category)**.** It is very common in category theory to form one category out of another category. Let's see an example of this. Let $\mathcal{C}$ be a category and X be an object in $\mathcal{C}$. Then, the *slice category* $\mathcal{C}/X$ is a category where:

- Objects are morphisms of $\mathcal{C}$ with codomain X, i.e. $O = \{A \xrightarrow{f} X \mid A \in \mathcal{C}\}$
- Morphisms are triples $(A \xrightarrow{f} X, A \xrightarrow{h} B, B \xrightarrow{g} X)$ satisfying $g \circ h = f$. These can be drawn as commutative triangles:

$$M = \left\{ \begin{array}{ccc} A & \xrightarrow{h} & B \\ & {}_{f}\searrow \quad \swarrow_{g} & \\ & X & \end{array} \;\middle|\; A \xrightarrow{h} B, A \xrightarrow{f} X, B \xrightarrow{g} X \text{are morphisms of } \mathcal{C}, f = g \circ h \right\}$$

Show that $\mathcal{C}/X$ is a category and identify the definition of dom, cod, id, and comp.

Problem 4 (STLC)**.** Recall the definition of STLC from Assignment 0. Contexts Γ were defined inductively using the following grammar:

$$\mathsf{Ctx} \ni \Gamma ::= \bullet \mid \Gamma, x : A$$

where $\bullet$ denotes the empty context. It is equivalently possible to view contexts as a finite product, i.e. the context $x_1 : A_1, \ldots, x_n : A_n$ viewed as a list gets interpreted as the product $A_1 \times \cdots \times A_n$ (where 1 is the empty product), and each variable x_i is interpreted as a projection $\mathsf{proj}_i : A_1 \times \cdots \times A_n \to A_i$. With this interpretation, STLC forms a category in the following sense:

- objects are types A;
- a morphism $A \to B$ is an equivalence class of STLC terms $[a : A \vdash M : B]$, given by:

$$[a : A \vdash M : B] = [a : A \vdash N : B] \iff a : A \vdash M \equiv N : B$$

- Composition is given by substitution:

$$[b : B \vdash N : C] \circ [a : A \vdash M : B] = [a : A \vdash N[M/b] : C]$$

- the identity morphism $\mathrm{id}_A : A \to A$ is given by the term $[a : A \vdash a : A]$.

Prove that STLC satisfies the axioms of a category. You will find yourself proving some utility lemmas about substitution, which are proven using structural induction. If you want, you can prove those lemmas for practice, or simply state them.

Problem 5. Show that Formula has products and a terminal object. What do they mean logically?

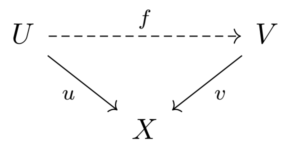
Problem 6. Show that $\mathsf{FinSet}^{\circlearrowright}$ has products. Does it have a terminal object? If you know what coproducts are: does it have coproducts?

Problem 7 (A rite of passage)**.** Suppose a category $\mathcal{C}$ has products. Prove that for any objects A, B, and C, it is the case that $(A \times B) \times C \cong A \times (B \times C)$ (recall that $\cong$ means isomorphism of objects).

Note: This may take a lot of space. We will see a better way to prove this later.

Problem 8. Let X be an object of FinSet. Show that elements of X (when considered as a finite set) are in bijection with morphisms in FinSet from 1 to X, where 1 is the terminal object of FinSet.

Problem 9. Suppose U and V are subsets of a finite set X. Show that $U \subseteq V$ if and only if there exists a dashed FinSet-morphism f such that $f \circ v = u$, where u and v are the FinSet-morphisms corresponding to the inclusion functions $U \to X$ and $V \to X$ respectively:



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